Transition to Advanced Mathematics

Five Problem-a-day Study Guide

Friday

1. Determine the truth value of the following statement: "Either 2 is rational and π is irrational, or 2π is irrational"
2. Prove that if x is odd and y is even, then $x + y$ is odd.
3. Prove the following statement using logic.
$$\big((P \Rightarrow Q) \wedge (R \Rightarrow S) \wedge (P \vee R)\big) \Rightarrow (Q \vee S)$$
4. Prove the following statement form is a tautology using a truth table.
$$\big((P \Rightarrow Q) \wedge (R \Rightarrow S) \wedge (P \vee R)\big) \Rightarrow (Q \vee S)$$
5. Let $x \in \mathbb{R}$ be greater than 6. Prove that:
$$\frac{x}{x+1} > \frac{x}{x+2}$$

Saturday

1. Write down an English sentence for the mathematical statement
$$\forall_{x\in\mathbb{R}}\exists_{y\in\mathbb{R}}(xy + y = 7)$$
2. Prove the following statement (Eratta: This statement is false)
$$\forall_{x\in\mathbb{R}}\exists_{y\in\mathbb{R}}(xy + y = 7)$$
3. Prove that $\exists_{\varepsilon>0}(\varepsilon^2 < \varepsilon)$
4. Prove that $\forall_{n>7}(n^3 > n^2)$
5. Find the negation of $\forall_{x\in\mathbb{R}}\exists_{y\in\mathbb{R}}(xy + y = 7)$.

Sunday

1. Prove that $\sqrt{17}$ is irrational.
2. Prove that the function below is one-to-one.
$$f\colon \mathbb{R} \to \mathbb{R}$$
$$x \mapsto 7x + 2$$
3. Prove that the function below is onto.
$$f\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$$
$$x \mapsto x^2$$
4. Let a and b be positive integers. Using the definition of $\gcd(a,b)$, show that
$$\gcd(a,b) \cdot lcm(a,m) = ab$$
5. Explain and show why $6|18$.

Monday (A, B, C, D are sets)

1. If $C \subseteq D$ and $D \subseteq B$, prove that $C \cap D \subseteq A \cap B$.
2. Prove that $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$
3. Prove that $A - (B - C) = (A - B) - C$
4. Prove that $(A \times B) \cap (B \times A) \subseteq (A \cap B) \times (A \cap B)$
5. How many elements will $\mathcal{P}(\mathcal{P}(\mathcal{P}(\{a\}))$ have? Find it.

Tuesday

1. Find both of the following

$$\bigcup_{m=1}^{\infty}\left(3 - \frac{1}{m}, m\right) \quad \text{and} \quad \bigcap_{m=1}^{\infty}\left(3 - \frac{1}{m}, m\right)$$

2. Prove:

$$\bigcap_{i \in I} A_i = \bigcup_{i \in I} A_i$$

3. Let I be an arbitrary index set and A_i sets indexed by I. Prove or disprove:

$$\left(\bigcup_{i \in I} A_i\right) - B = \bigcup_{i \in I}(A_i - B)$$

4. Suppose that $J \subseteq I$ are index sets. Fill in each of the boxes below with either "$=$", "$\subseteq$", or "$\supseteq$". Then prove them.

$$\bigcap_{i \in J} A_i \square \bigcap_{i \in I} A_i \quad \text{and} \quad \bigcup_{i \in J} A_i \square \bigcup_{i \in I} A_i$$

5. Prove for all integers $n \geq 2$, that:

$$\left(\bigcup_{m=1}^{n} A_m\right)^c = \bigcap_{m=1}^{n} A_m^c$$

Wednesday

1. Let X be a set with n elements. Prove that $|\mathcal{P}(X)| = 2^n$.
2. Figure out what the ??s should be, then Prove that $\sum_{m=1}^{n}(2m-1) = (??)^2$ for all $n \geq 2$.
3. Use induction to prove that:
$$\sum_{m=1}^{n} \frac{1}{(2m-1)(2m+1)} = \frac{n}{2n+1}$$
4. Let $r \neq 1$. Prove that:
$$\sum_{m=1}^{n} r^m = \frac{r^{n+1}-1}{r-1}$$
5. Let $r \neq 1$. Prove that:
$$\sum_{m=1}^{n} mr^m < \frac{r}{(1-r)^2}$$

Thursday

1. Prove that $\sum_{m=1}^{n} m^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \geq 0$.
2. Prove that $\sum_{m=1}^{n}(-1)^{m-1}m^2 = \frac{(-1)^{n-1}n(n+1)}{2}$ for all $n \geq 1$
3. Define a relation R on $\mathbb{Z}^2$ via $(a,b)R(x,y)$ if and only if $a \equiv_4 x$ and $b \equiv_5 y$. Prove or disprove that R is an equivalence relation.
4. Define a relation “Mod 3 or Mod 7” via aRb if and only if $a \equiv_3 b$ or $a \equiv_7 b$. Prove or disprove that R is an equivalence relation.
5. Give an example of a relation that is reflexive, antisymmetric, but not transitive.

Friday

1. Find 5^{-1} mod 26.
2. Prove that subtraction is well defined in mods. First, define exactly what "$\bar{x} - \bar{y}$" means, where the relation is mod m. This definition depends on the representatives x and y you chose – show that the solution is the same, regardless.
3. Solve $17x^2 + 4 \equiv 32$ mod 50
4. Figure out what numbers have multiplicative inverses in mod 12. Conjecture a theorem that might work for other mods.
5. We'll construct the following set: $\mathbb{R}/\mathbb{Z}$ in this problem.
    a. The relation $\equiv$ will be defined via xRy if and only if $x - y \in \mathbb{Z}$. Prove that R is an equivalence relation. Then we may start using the notation "$\equiv$" for R.
    b. Find the equivalence class $\bar{3}$.
    c. Find the equivalence class $\overline{3.2}$.
    d. Write down the set of all equivalence classes.
    e. Construct a bijection between the set of all equivalence classes, and $[0,1)$.

Saturday

1. Show that the function f, below, is one-to-one.
$$f: \mathbb{R} \to \mathbb{R}^2$$
$$x \mapsto (x^2, x^3)$$
2. Show that the function, f, below, is onto.
$$f: \mathbb{R}^3 \to \mathbb{R}^2$$
$$(x, y, z) \mapsto (x^3 z, y^2 z)$$
3. Show that the function $f|_{\mathbb{R}\times\{0\}}$ is one-to-one, where f is:
$$f: \mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \mapsto x^2 y + 2x + 3y + 4$$
4. Let $f: A \to B$ be a function and $S \subseteq B$. The pre-image of S is defined as $f^{-1}(S) \coloneqq \{a \in A | f(a) \in S)$. Given f below, find $f^{-1}([4,7])$.
$$f: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^2$$
5. Find f^{-1}, where f is given below. Hint: You might have to write out a table of values for f^{-1}.
$$f: \mathbb{Z}_{11} \to \mathbb{Z}_{11}$$
$$x \mapsto x^3$$

Sunday

1. The function f, below, is not invertible. Define the largest possible restriction, $g \coloneqq f|_S$ such that g is invertible. Then find the rule that defines g.
2. Prove that the function f, below, is never invertible – no matter what positive integer m is.

$$f: \mathbb{Z}_m \to \mathbb{Z}_m$$
$$x \mapsto x^2$$

3. For every positive integer n, find a sequence of distinct functions $f_1, f_2, f_3, \dots, f_n$ such that:

$$f_1 \circ f_2 \circ f_3 \circ \cdots \circ f_n = f$$

where f is given below.

$$f: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto 2x + 3$$

4. Find f^{-1} for f given below, then prove that f^{-1} is the inverse.

$$f: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto 2x + 3$$

5. Prove that the following statement is true: "I will get a good night sleep tonight"